Dynamic Analysis Using a Reduced Basis of Exact Modes and Ritz Vectors

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The problem of obtaining an approximate solution for the response of a structure under dynamic forces is addressed. The structure is modeled using mass and stiffness matrices and has n degrees of freedom. An m-dimensional approximation is sought in a subspace defined by an arbitrary M-orthogonal array Σ. A residual vector, which is smaller in magnitude the more accurate the approximation, is derived. The residual contains two terms: one is the error involved representing the force vector in the Σ subspace; the other is smaller the better the Σ array leads to m approximate solutions of the exact eigenproblem. Numerical experiments are performed using Σ arrays that are combinations of exact eigenvectors and of Ritz vectors. In one case, when 11 different 10-dimensional bases are used to approximate the solution of a 60 degree of freedom problem, it is found that a root-mean-square measure of relative error is smallest for the basis formed using the 6 lowest exact eigenvectors plus 4 Ritz vectors generated from the spatial distribution of the loading.

Introduction

IN considering m-dimensional approximations to n-dimensional problems in which a structure is characterized by mass and stiffness matrices and for which the deformed configuration is sought when the structure is subjected to time-dependent forces, Wilson et al.[1] and Bayo and Wilson[2] recently presented numerical results showing that for some problems more accurate approximations can be obtained when an m-dimensional basis is formed using appropriately chosen Ritz vectors than are obtained when the lowest m exact modes are used for the basis. They note that Ritz methods have been used extensively in structural dynamics, with one motivation being the desire to avoid the cost of an exact eigenvalue problem.

As discussed by Bathe,[3] Ritz methods are used in procedures to determine the m lowest eigenvalues and corresponding eigenvectors of a structure; one generates a basis using the m static solutions resulting from m load patterns. From any one starting vector, one can generate additional vectors for the basis using the Lanczos[4] method. Meirovitch and Hale[5] recently observed that the substructure synthesis method can be thought of conveniently in the context of Ritz analysis.

In procedures to obtain the m lowest exact eigenvalues and eigenvectors[3-5] and, also in efforts to use Ritz vectors as a basis when displacements due to time-dependent forces are sought,[1,2] an open issue is whether the Ritz vectors taken will lead to accurate representation of the exact lowest eigenvalues and eigenvectors. It is certainly true that, as stated in Ref. 1, no proof exists that the use of m exact eigenvectors in mode superposition analysis is better than the use of any other set of orthogonal vectors. (Indeed, the error estimate developed in the present work would provide counterexamples.) However, it is also true that one cannot be assured that a basis formed solely from Ritz vectors will be better in mode superposition analysis than another orthogonal set of vectors.

In view of the facts stated above, our starting point in seeking an m-dimensional approximation to an n-dimensional problem is to introduce the transformation

$$\tilde{x}=\Sigma v(t)$$

where Σ is an arbitrary M-orthogonal $n\times m$ array, and $\tilde{x}$ is the resulting n-dimensional approximate displacement vector. We form the residual vector

$$r(t)=f(t)-M\ddot{\tilde{x}}-K\tilde{x}$$

where $f(t)$ is the forcing function, and show that $r(t)$ is simply the sum of the two following terms [see Eq. (26)]:

1) A term representing the error involved in representing f in the Σ subspace.

2) A term involving the extent to which Σ will lead to m solutions of the exact eigenproblem for K and M.

This result motivates considering a starting basis Σ composed of some number of exact eigenvectors plus some Ritz vectors. (In numerical results, we consider higher-order generations of Ritz vectors from one starting vector, in the spirit of Lanczos.[1,2,4]) We simply ask if, given some exact eigenvectors, there is an efficient procedure for improving the accuracy of dynamic analysis. We find there is—it appears that augmenting exact eigenvectors by Ritz vectors leads to substantial reductions in errors. Further, this is inherently efficient since the Ritz vectors are generated by triangularizing the stiffness matrix,[1,2] an operation that likely has already been performed in determining stresses due to static loads.

In particular, the present work is motivated by problems involving large, complex automotive and aerospace structures. Typically, in such analyses, some type of component mode synthesis[3] is performed and certain of the lowest exact eigenvalues and eigenvectors of the components and full systems are calculated. One reason for this is that experimental results are often available, in which case efforts can be made to improve the analytical models by comparing finite element predictions with experimentally determined natural frequencies and mode shapes. What we demonstrate is that even in situations where one has confidence in the finite element model predictions of natural frequencies and mode shapes, it is still advisable to augment the exact eigenvectors by suitably chosen Ritz vectors when developing a reduced-dimensional basis for analysis of structural response under action of time-dependent forcing vectors. This work is thus a rather natural extension of previous studies[6,7] directed toward improving mode superposition analysis.

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Mode Superposition Analysis

In this section, we consider the task of obtaining an approximate solution to an n-degree-of-freedom system using the method of mode superposition. In general terms, we ask if, given an n-dimensional problem, there is some best m-dimensional subspace to use in approximate analysis. By examination of a residual vector, we conclude that an improved basis may well be formed from a combination of exact eigenvectors and appropriately determined Ritz vectors. Numerical experiments that support this conclusion are reported in the next section.

To set the problem, let M and K denote mass and stiffness matrices, $x(t)$ the unknown displacement vector, and $f(t)$ the loading vector. Neglecting damping, although observing Rayleigh damping can easily be handled, we have

$$M\ddot{x}+Kx=f(t) \tag{1}$$

Suppose we seek an m-dimensional approximation of the form

$$\tilde{x}=\Sigma v(t) \tag{2}$$

where Σ represents an $n\times m$ array that, without loss of generality, we can M-orthogonalize as

$$\Sigma^T M\Sigma=I_{(m)} \tag{3}$$

where $I_{(m)}$ denotes the m-dimensional identity matrix.

Substituting Eq. (2) into Eq. (1) and defining a reduced m by m stiffness

$$K_R=\Sigma^T K\Sigma \tag{4}$$

we obtain

$$\ddot{v}+K_R v=\Sigma^T f \tag{5}$$

Although K_R is not in general a diagonal matrix, it is symmetric and we can solve the reduced problem of Eq. (5) exactly using mode superposition with the exact eigenvectors. Let Φ^* and Λ^* denote the eigenvector and eigenvalue arrays related to K_R,

$$K_R\Phi^*=\Phi^*\Lambda^* \qquad \Phi^{*T}\Phi^*=I_{(m)} \tag{6}$$

Then making the expansion

$$v=\Phi^* w \tag{7}$$

produces the uncoupled equations

$$\ddot{w}+\Lambda^* w=(\Phi^{\#})^T f \tag{8}$$

where $\Phi^{\#}$ is an $n\times m$ array,

$$\Phi^{\#}=\Sigma\Phi^* \tag{9}$$

From Eqs. (3), (4), (6), and (9), we obtain

$$(\Phi^{\#})^T M\Phi^{\#}=I_{(m)} \qquad (\Phi^{\#})^T K\Phi^{\#}=\Lambda^* \tag{10}$$

Further, combining Eqs. (2) and (7) yields

$$\tilde{x}=\Phi^{\#} w \tag{11}$$

which, through Eq. (8), is the form of approximate solution sought for Eq. (1).

If $\Phi^{\#}$ contained m eigenvectors of K and M and Λ^* contained the corresponding m eigenvalues, then Eq. (10) would be satisfied. However, for $m<n$, Eq. (10) implies neither that $\Phi^{\#}$ contains eigenvectors nor that Λ^* contains eigenvalues.

To investigate the nature of the approximation, we define the residual vector $r(t)$

$$r(t)=f(t)-M\ddot{\tilde{x}}-K\tilde{x} \tag{12}$$

Substitution of Eq. (11) into Eq. (12) yields

$$r(t)=f(t)-M\Phi^{\#}\ddot{w}-K\Phi^{\#} w \tag{13}$$

which, with Eq. (8), produces

$$r=[I-M\Phi^{\#}(\Phi^{\#})^T]f-[K\Phi^{\#}-M\Phi^{\#}\Lambda^*]w \tag{14}$$

Note the entire second term of r vanishes if

$$K\Phi^{\#}=M\Phi^{\#}\Lambda^* \tag{15}$$

and that this condition would simply mean $\Phi^{\#}$ and Λ^* contain m solutions to the n-dimensional eigenproblem. Also, using Eq. (10), we obtain

$$(\Phi^{\#})^T r=0 \tag{16}$$

which confirms that r is orthogonal to the column space of $\Phi^{\#}$. Finally, the fact that the behavior of the first term of Eq. (14) cannot be described without specifying the load vector f suggests that for a given f an appropriate set of basis vectors $\Phi^{\#}$ may well include Ritz vectors generated using f. Indeed, consider approximating f in the $\Phi^{\#}$ subspace

$$f=M\Phi^{\#} q+e \tag{17}$$

where e is the "error" of the approximation. Using Eq. (10)

$$(\Phi^{\#})^T f=q \tag{18}$$

since we must have $(\Phi^{\#})^T e=0$.

From Eqs. (17) and (18)

$$e=[I-M\Phi^{\#}(\Phi^{\#})^T]f \tag{19}$$

which is the first term of the residual r of Eq. (14).

However, the starting point of our reduction to an m-dimensional problem [Eq. (2)] involves the $n\times m$ array Σ; it is with respect to this array that one would attempt to incorporate Ritz vectors. To introduce Σ in the residual vector $r(t)$ of Eq. (14) we note that, using Eqs. (9) and (6)

$$\Phi^{\#}(\Phi^{\#})^T=\Sigma\Sigma^T \tag{20}$$

which allows us to write

$$r=[I-M\Sigma\Sigma^T]f-[K\Phi^{\#}-M\Phi^{\#}\Lambda^*]w \tag{21}$$

Using Eq. (3)

$$\begin{aligned}\Sigma^T r&=-\Sigma^T[K\Phi^{\#}-M\Phi^{\#}\Lambda^*]w\\&=-\Sigma^T[K\Sigma\Phi^*-M\Sigma\Phi^*\Lambda^*]w\\&=-[K_R\Phi^*-\Phi^*\Lambda^*]w\\&=0\end{aligned} \tag{22}$$

where the second equality follows from Eq. (9), the third from Eqs. (3) and (4), and the last from Eq. (6). Hence, although the approximate solution to Eq. (1) now takes the

form of Eq. (11) and is complete on solution of Eq. (8), and although Eqs. (8) and (11) involve the array $\Phi^{\#}$, we see that the residual vector r of Eq. (14) is in fact orthogonal to the column space of Σ and recall that Σ is, through Eq. (2), the starting point of the approximate analysis.

Force vector f may be represented also in the Σ subspace. Rather than Eq. (17), write

$$f = M\Sigma q' + e' \tag{23}$$

where e' is the error of the approximation, Eq. (23). Since $\Sigma^T e' = 0$, by Eq. (3)

$$\Sigma^T f = q' \tag{24}$$

and thus from Eqs. (23), (18), and (19)

$$e' = [I - M\Sigma\Sigma^T]f = e \qquad q = \Phi^{*T} q' \tag{25}$$

Finally, combining Eqs. (21) and (25), we obtain an enlightening form for the residual vector of Eq. (12)

$$\begin{aligned} r &= f(t) - M\ddot{\tilde{x}} - K\tilde{x} \\ &= e' - [K\Phi^{\#} - M\Phi^{\#}\Lambda^*]w \end{aligned} \tag{26}$$

from which we observe that e' is simply the error [Eq. (23)] inherent in representing force vector f in the Σ subspace and that the second term is in general smaller, the better the $\Phi^{\#}$ and Λ^* arrays represent eigensolutions related to matrices K and M.

Now, our analysis objective is to achieve the best m-dimensional approximation to Eq. (1) for *specific design loads* $f(t)$. One can imagine using various design loads to generate Ritz basis vectors[3] for inclusion in subspace Σ. This would reduce error component e', but would not necessarily produce arrays $\Phi^{\#}$ and Λ^* that approximate well eigensolutions to K and M. Indeed, even if the Lanczos method[3,4] is used to generate a family of basis vectors for each design load, one cannot be assured that good approximations to the eigenvalues and eigenvectors of K and M will result, a fact demonstrated by numerical experiments reported in the next section.

On the other hand, reliance entirely on a number m of the lowest exact modes of K and M does not guarantee that the resulting subspace provides a good approximation to force vector $f(t)$. Thus the error e' may be large, a fact demonstrated by numerical experiments of Wilson and his colleagues.[1,2]

As described in the next section, an improved basis may be formed by combining exact eigenvectors with appropriate Ritz vectors.

Algorithm and Numerical Experiments

Our algorithm is in the spirit of that given by Wilson et al.[1] and Bayo and Wilson.[2] We use the parameters: NNM—number of normal modes (exact eigenvectors) and NGR—number of Ritz vectors generated for each load case.

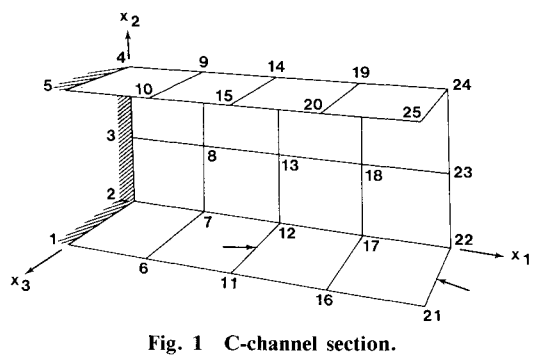

Fig. 1 C-channel section.

The $n \times m$ array Σ is, in partitioned form

$$\Sigma = [\Phi \;\; \Psi] \tag{27}$$

where Φ contains the NNM lowest eigenvectors and ψ is an array with NGR columns, denoting that NGR Ritz vectors are to be generated for each load case. Since there are m columns of Σ, $m = \text{NNM} + \text{NGR}$.

The M-orthogonality of Eq. (3) is developed in the standard fashion.[4] Consider one force vector f. A solution vector is obtained from

$$K\psi_1^* = f \tag{28}$$

Then, define a vector M-orthogonal to array Φ

$$\begin{aligned} \psi_1^{**} &= \psi_1^* - \Phi c \\ c &= \Phi^T M \psi_1^* \end{aligned} \tag{29}$$

Finally, normalize by

$$\psi_1 = \psi_1^{**} / [(\psi_1^{**})^T M \psi_1^{**}]^{1/2} \tag{30}$$

The second generation Ritz vector builds from

$$K\psi_2^* = M\psi_1 \tag{31}$$

with the augmented array $[\Phi \;\; \psi_1]$ used in place of Φ in Eq. (29)

$$\begin{aligned} \psi_2^{**} &= \psi_2^* - [\Phi \;\; \psi_1]c \\ c &= [\Phi \;\; \psi_1]^T M \psi_2^* \end{aligned} \tag{32}$$

and normalization following similar to Eq. (30). One continues this process through NGR generations of Ritz vectors until the $n \times m$ array Σ is obtained. The analysis then follows the steps of the previous section, with the m-dimensional problem solved exactly for w [Eq. (8)] and the approximate solution obtained from Eq. (11).

Numerical experiments were performed on the C-channel section shown in Fig. 1. This structure was modeled using MSC/NASTRAN with 16 CQUAD4 elements and 8 CBAR elements, the latter representing stiffeners, with 4 each along the free end and the midline (the sections marked by arrows). The structure had 60 translational degrees of freedom and 48 rotational (massless) degrees of freedom and mass lumping was used. The analysis set was taken to include only the 60 translational degrees of freedom. Thus, in this case, the full problem had $n = 60$, sufficiently small so that we could easily calculate the exact solution vector x.

Letting $(x - \tilde{x})_i$ and $(x)_i$ represent the ith components of the respective vectors, we define a rms measure of relative error as

$$\text{rms} = \left\{ \sum_{i=1}^{60} \left[\frac{(x - \tilde{x})_i}{(x)_i} \right]^2 \right\}^{1/2} \tag{33}$$

In all of the numerical experiments discussed below, exact solutions and m-dimensional approximate solutions were obtained for frequency response, for which

$$f(t) = f_0 \sin\omega t; \qquad \lambda = \omega^2 \tag{34}$$

In Table 1 we present results for the C-channel section in frequency response, with $\lambda = \omega^2 = 9.0 \times 10^5$ (rad/s)^2. In all cases, 10-dimensional subspaces were used to obtain approximate solutions. The rms measure was found using Eq. (33). The force acted at node 20 and had equal components in all three coordinate directions. Observe that, for this loading,

the rms error is smallest for the 10-dimensional basis formed using 6 exact modes and 4 Ritz vectors.

Results for single point loading at node 20 for two forcing frequencies are shown in Table 2. Note that a 10-dimensional subspace formed using 9 exact modes and 1 Ritz vector yields more accurate approximations than those obtained using 15 exact modes. Also observe the addition of just 1 Ritz vector to either 10 or 15 exact modes leads to substantial improvements in accuracy.

To investigate the meaning of these numerical results, we consider the second term in the residual vector of Eq. (26). Approximate eigenvalues calculated in the m-dimensional reduced space [Eq. (6)] are compared with the first 16 exact eigenvalues in Table 3. Observe that in the 10 modes/6 Ritz vectors case the estimates for eigenvalues 14, 15, and 16 are poor and further that the 16th eigenvalue is not well approximated using the 15 modes/1 Ritz vector subspace.

In Table 4 we show results for two 16-dimensional approximations: one with 15 modes plus 1 Ritz vector, the other with 10 modes and 6 Ritz vectors. The results for the former subspace are substantially better when the forcing function acts at node 20 and is harmonic, with $\lambda = 3.0 \times 10^6$ (rad/s)^2. Neither 16-dimensional subspace produces particularly good results when the forcing frequency is $\lambda = 2.0 \times 10^8$ (rad/s)^2, a frequency that turns out to be between exact eigenvalues 22 (1.5168×10^8) and 23 (2.0117×10^8). In fact, the rms error increases in the 17-dimensional space formed using 15 modes and 2 Ritz vectors! To see how this could occur, we consider the residual vector for frequency response. When f is given by Eq. (34), then Eq. (26) produces

$$r = e' - (K\Phi^{\#} - M\Phi^{\#}\Lambda^*)(\Lambda^* - \lambda I)^{-1}(\Phi^{\#})^T f \qquad (35)$$

When the forcing frequency is $\lambda = 2.0 \times 10^8$, a 17-dimensional approximation is inadequate. For the case examined, the 16th eigenvalue was predicted quite well (2.550×10^7 vs the exact value of 2.305×10^7), whereas the 17th eigenvalue was poorly estimated (1.853×10^8 vs the exact value of 2.807×10^7). Hence, the $n \times m$ (60×17) array $(K\Phi^{\#} - M\Phi^{\#}\Lambda^*)$ has a "large" 17th column that multiplies a "large" factor, the 17th diagonal entry of $(\Lambda^* - \lambda I)^{-1}$. In turn, this causes a larger residual for the 17-dimensional approximation than for the 16-dimensional although e' is reduced, i.e., even though f_0 is better described in the larger subspace.

Hence, it is necessary for the starting Σ array to lead to good approximation of the eigenproblem in the frequency range of the loading to be certain the error will be small, even if the basis handles the spatial distribution of the loading well. Ritz vectors by themselves may not be enough.

This is further illustrated by numerical experiments in which we attempted to "rank order" exact modes for inclusion in Σ. In particular, seeking a "best" 10-dimensional approximation, we ordered the lowest 10 modes according to the magnitude of $\phi \cdot f_0$. The idea was to extract the "most active" modes so that, for example, with 7 modes plus 3 Ritz vectors, the 7 modes would not necessarily be the 7 lowest, but rather the 7 most active. The result was a disaster. With forcing frequency $\lambda = 9.0 \times 10^5$, the rms error for the 7 lowest modes plus 3 Ritz vectors was 0.031; using the 7 most active of the first 10 exact modes, plus 3 Ritz vectors, led to an rms error of 12.4! The explanation is clear using Eq. (35). The exact fourth eigenvalue, which is recovered in the "lowest" 7 plus 3 case, was 8.386×10^5. The predicted fourth eigenvalue when the seven "most active" modes (which did not include exact mode 4) were used was 8.716×10^5; this error in representing the eigensolution led to a large factor in $(K\Phi^{\#} - M\Phi^{\#}\Lambda^*)$, which in turn multiplied a factor $(\lambda^* - \lambda)^{-1}$ that was larger than if the exact fourth eigenvalue was recovered.

In Table 5 are shown results for three 12-dimensional approximations when two forcing frequencies that are close to natural frequencies are applied: $\lambda = 1.0 \times 10^6$ is 0.25% greater than the fifth exact eigenvalue (given in Table 3); $\lambda = 2.2 \times 10^6$ is 0.99% less than the tenth exact eigenvalue. As before, the force acts at node 20 with equal components in all 3 coordinate directions. Consistent with previous results, the error is reduced when Ritz vectors are used. The reason is that in these cases the starting Σ array contains 10 exact eigenvectors that provide a good approximation to the exact eigenproblem in the frequency range of the loading and, hence, do not allow the second term in the expression for the residual vector [Eq. (25)] to become large. The issue then is, with 10 exact modes in Σ, to find 2 additional vectors that will lead to the smallest rms error. Of the three choices examined, the best option is to use two Ritz vectors so as to reduce the first term e' in the residual [Eq. (35)], i.e., use vectors that help better to describe the spatial distribution of loading f_0 in the approximation space.

However, as discussed with reference to Table 4, the residual vector contains two separate contributing terms. If the Σ array does not allow rather accurate approximation of exact eigenvalues within neighborhoods of forcing frequen-

Table 1 Displacement errors (rms) using 10-dimensional subspaces for approximations

No. modes	No. Ritz vectors	rms error
10	0	1.91
9	1	0.543
8	2	0.078
7	3	0.031
6	4	0.018
5	5	0.020
4	6	0.155
3	7	2.99
2	8	8.63
1	9	20.5
0	10	41.2

Table 2 Displacement errors (rms) using various m-dimensional subspaces for approximations, for two loading frequencies

Dimen. subspace	No. modes	No. Ritz vectors	rms ($\lambda = 2.6 \times 10^5$)	rms ($\lambda = 9.0 \times 10^5$)
10	10	0	2.96	1.91
15	15	0	2.69	0.849
11	10	1	0.189	0.356
16	15	1	0.006	0.008
10	9	1	0.281	0.543

Table 3 Approximate and exact eigenvalues

Exact	NNM = 15 NGR = 1	NNM = 10 NGR = 6
2.850603E + 05	2.850603E + 05	2.850603E + 05
4.097885E + 05	4.097885E + 05	4.097885E + 05
5.229374E + 05	5.229374E + 05	5.229374E + 05
8.385665E + 05	8.385665E + 05	8.385665E + 05
9.975357E + 05	9.975357E + 05	9.975357E + 05
1.268315E + 06	1.268315E + 06	1.268315E + 06
1.431666E + 06	1.431666E + 06	1.431666E + 06
1.707274E + 06	1.707274E + 06	1.707274E + 06
2.189083E + 06	2.189083E + 06	2.189083E + 06
2.221961E + 06	2.221961E + 06	2.221961E + 06
2.700899E + 06	2.700899E + 06	2.700901E + 06
3.332824E + 06	3.332824E + 06	3.332835E + 06
5.825157E + 06	5.825157E + 06	5.874616E + 06
6.328079E + 06	6.328079E + 06	1.160336E + 07
9.305816E + 06	9.305816E + 06	2.842051E + 07
2.305234E + 07	4.233172E + 07	2.899529E + 08

Table 4 Displacement errors for four loading frequencies using m-dimensional approximations

Vectors		rms			
Modes	Ritz	$\lambda = 2.6E5$	9.0E5	3.0E6	2.0E8
15	1	6.122×10^{-3}	7.965×10^{-3}	3.467×10^{-2}	1.874×10^{1}
10	6	1.113×10^{-9}	9.392×10^{-8}	1.090×10^{-1}	8.512×10^{1}
15	0	2.688	8.490×10^{-1}	4.485×10^{-1}	1.533×10^{1}
15	2	1.296×10^{-5}	6.869×10^{-5}	3.200×10^{-3}	9.656×10^{1}

Table 5 Displacement errors for forcing frequencies near natural frequencies using 12-dimensional approximations

Vectors		rms	
Modes	Ritz	$\lambda = 1.0E6$	$\lambda = 2.2E6$
10	2	9.291×10^{-4}	2.475×10^{-2}
11	1	1.729×10^{-2}	1.329×10^{-1}
12	0	1.071×10^{-1}	2.112×10^{-1}

cies, the second term of Eq. (35) can become large and, even if the first term vanishes, a poor approximate solution may result. For example, with loading frequency $\lambda = 3.0\times10^{6}$, which is about midway between the 11th and 12th exact eigenvalues, the 12-dimensional approximation formed with the lowest 10 modes and 2 Ritz vectors yields an rms error of 1.083×10^{2}, which is 3 orders of magnitude larger than the corresponding rms error (shown in Table 4) arising from the approximation based on the 15 lowest exact modes.

Let us extend this and consider a practical situation. Suppose the frequency content of a design load involves λ values up to about 3.0×10^{6}. Further, suppose that experimental tests on actual hardware yield 15 natural frequencies corresponding to λ values less than 10.0×10^{6} and that predictions from an analytical model agree rather well with these 15 experimental frequencies. Then the numerical experiments summarized in Tables 2 and 4 would suggest using the lowest 15 modes of the analytical model plus a Ritz vector or two to predict response of the structure when loaded. Ten modes plus six Ritz vectors or some other combination might actually be better, but the question of whether to include in the approximation space the 16th mode rather than a Ritz vector seems to have been resolved: numerical experiments suggest including a Ritz vector. Theoretical results of the next section lend support to this.

The Residual Vector

The expression for the residual vector r given by Eq. (26) is valid for an arbitrary M-orthogonal array Σ and an arbitrary force vector $f(t)$. When the force is specialized to the case of frequency response [Eq. (34)], the residual vector for arbitrary Σ takes the form of Eq. (35).

Motivated by the numerical results of the previous section, we now turn to examination of the residual vector for the case when Σ is composed of q exact eigenvectors and p additional vectors. Thus, consider

$$\Sigma = [\Phi \ \ \Psi] \tag{36}$$

where

$$\Phi = [\phi_1 \phi_2 \ldots \phi_q] \tag{37}$$

$$\Psi = [\psi_1 \psi_2 \ldots \psi_p] \tag{38}$$

The results established in this section are valid when the columns of the $n \times q$ array Φ [Eq. (37)], represent any q eigenvectors associated with stiffness matrix K and mass matrix M. A reasonable choice would be the eigenvectors associated with the q lowest eigenvalues of K and M. However, we here assume only that we have some q eigensolutions that admit the form

$$K\Phi = M\Phi\Lambda \tag{39}$$

where the diagonal array Λ contains the q eigenvalues associated with the q eigenvectors in Φ

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \vdots & \\ & & & \lambda_q \end{bmatrix} \tag{40}$$

In a departure from the previous section, the columns of array Ψ are not assumed here necessarily to result from Ritz vectors; i.e., we do not require Ψ_1^* to satisfy Eq. (28) or Ψ_2^* to satisfy Eq. (31). Rather, we regard Ψ_1^* as a starting vector and satisfy the M-orthogonality by applying Eq. (29) to generate Ψ_1^{**}. Then, one must normalize, using Eq. (30) to obtain ψ_1. (If Ψ_1^* is a linear combination of the vectors in Φ, then Ψ_1^{**} is a zero vector. Thus, in practice, one must monitor this type of Gram-Schmidt procedure to ensure that null vectors are not generated and that M-orthogonality is preserved.) To generalize this process, we define the array

$$\Psi^{(k)} = [\Psi_1 \Psi_2 \ldots \Psi_k] \tag{41}$$

Then Ψ_{k+1}^{**} is to be obtained from Ψ_{k+1}^{*} by

$$\Psi_{k+1}^{**} = \Psi_{k+1}^{*} - [\Phi \ \ \Psi^{(k)}]\,[\Phi \ \ \Psi^{(k)}]^T M \Psi_{k+1}^{*} \tag{42}$$

We first prove that, with Φ and Ψ defined by Eqs. (37) and (38),

$$\Phi^T K \Psi = 0 \tag{43}$$

A proof by induction goes easily. Taking $p = 1$ in Eq. (38), we are to show

$$\Phi^T K \Psi_1 = 0 \tag{44}$$

However, Eq. (44) is true if

$$\Phi^T K \Psi_1^{**} = 0 \tag{45}$$

By Eq. (29)

$$\Phi^T K \Psi_1^{**} = \Phi^T K \{\Psi_1^{*} - \Phi\Phi^T M \Psi_1^{*}\} \tag{46}$$

Now, since we have q eigensolutions

$$\Phi^T K \Phi = \Lambda \qquad \Phi^T M \Phi = I \tag{47}$$

Using Eq. (47) in Eq. (46) yields

$$\Phi^T K \Psi_1^{**} = [\Phi^T K - \Lambda \Phi^T M] \Psi_1^{*} \tag{48}$$

Now, the transpose of Eq. (39) shows

$$\Phi^T K = \Lambda \Phi^T M \tag{49}$$

and thus the right-hand side of Eq. (48) vanishes to establish Eq. (45) and thus Eq. (44). We now assume Eq. (43) is valid for $p = k$,

$$\Phi^T K \Psi^{(k)} = 0 \tag{50}$$

We are to prove the result for $p = k+1$, namely,

$$\Phi^T K [\Psi^{(k)} \ \ \Psi_{k+1}] = 0 \tag{51}$$

Expanding the left-hand side of Eq. (51) by using Eq. (50),

$$\Phi^T K [\Psi^{(k)} \ \ \Psi_{k+1}] = [0 \ \ \Phi^T K \Psi_{k+1}] \tag{52}$$

Thus, Eq. (51) is true if

$$\Phi^T K \Psi^{**}_{k+1} = 0 \tag{53}$$

Using Eqs. (42) and (47),

$$\Phi^T K \Psi^{**}_{k+1} = \Phi^T K \Psi^{*}_{k+1} - [\Lambda \ \ \Phi^T K \Psi^{(k)}] \{\tfrac{\Phi^T}{\Psi^{(k)T}}\} M \Psi^{*}_{k+1}$$

Using Eq. (50), this is

$$\Phi^T K \Psi^{**}_{k+1} = \Phi^T K \Psi^{*}_{k+1} - \Lambda \Phi^T M \Psi^{*}_{k+1}$$

which vanishes by virtue of Eq. (49) to establish Eq. (51) and complete the proof of Eq. (43).

There are two immediate corollaries. First, with array Σ given by Eq. (36), the reduced stiffness matrix is block diagonal,

$$K_R = \begin{bmatrix} \Lambda & 0 \\ 0 & \Psi^T K \Psi \end{bmatrix} \tag{54}$$

[Equation (54) follows by application of the definition of Eq. (4) and Eqs. (43) and (47).] Second, q of the $m = q+p$ eigenvalues of K_R are simply the exact eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_q$ given in Λ [Eq. (40)].

The simplicity of the result in Eq. (54) invites the following quest. Say we have a q-dimensional approximation due to an array Σ composed of q eigenvectors. We ask if, for a specific force $f(t)$, we can construct a vector ψ so as to produce a $(q+1)$-dimensional approximation that reduces the magnitude of the residual. To fix the notation, let $r^{(q)}$ be the residual when Σ is composed entirely of q eigenvectors. Then, only the first term of Eq. (26) contributes and from Eq. (25) we obtain

$$r^{(q)} = P^T f \tag{55}$$

where

$$P = I - \Phi \Phi^T M; \qquad P^T = I - M \Phi \Phi^T \tag{56}$$

Now let $r^{(q+1)}$ denote the residual when

$$\Sigma = [\Phi \ \ \psi] \tag{57}$$

We write

$$r^{(q+1)} = r^{(q)} + \Delta r \tag{58}$$

Our objective is, given $r^{(q)}$, to find that ψ which causes $r^{(q+1)}$ to be as small in magnitude as possible. Applying Eqs. (25) and (26) for Σ given by Eq. (57) yields

$$r^{(q+1)} = [I - M\Phi\Phi^T - M\psi\psi^T] f - [K\Phi^{\#} - M\Phi^{\#}\Lambda^*] w \tag{59}$$

where w satisfies Eq. (8), $\Phi^{\#}$ is given by Eq. (9) and Φ^* and Λ^* contain the $m = q+1$ eigenvectors and eigenvalues of K_R; see Eq. (6).

Now Σ given by Eq. (57) corresponds to the case $p = 1$ in Eq. (36) and, thus, in this case, by Eq. (54)

$$K_R = \begin{bmatrix} \Lambda & 0 \\ 0 & \psi^T K \psi \end{bmatrix} = \Lambda^* \tag{60}$$

That is, K_R is diagonal (and hence equal to its diagonal array of eigenvalues) if Σ is given by Eq. (57). From Eq. (60), the eigenvector array then becomes the identity matrix, $\Phi^* = I_{(q+1)}$ and, using Eq. (39),

$$\begin{aligned} K\Phi^{\#} - M\Phi^{\#}\Lambda^* &= K\Sigma - M\Sigma\Lambda^* \\ &= K[\Phi \ \ \psi] - M[\Phi \ \ \psi] \begin{bmatrix} \Lambda & 0 \\ 0 & \psi^T K \psi \end{bmatrix} \\ &= [0 \ \ (K\psi - M\psi\lambda^*)] \end{aligned} \tag{61}$$

where

$$\lambda^* = \psi^T K \psi \tag{62}$$

By Eq. (61) the first q columns of $[K\Phi^{\#} - M\Phi^{\#}\Lambda^*]$ are zero vectors. Thus, denoting components of w by

$$w^T = [w_1, w_2, \ldots, w_q, w_{q+1}] \tag{63}$$

we obtain, using Eqs. (55), (56), (59), (61), and (63)

$$\begin{aligned} \Delta r &= r^{(q+1)} - r^{(q)} \\ &= -M\psi\psi^T f - (K\psi - M\psi\lambda^*) w_{q+1} \end{aligned} \tag{64}$$

where, by Eq. (8)

$$\ddot{w}_{q+1} + \lambda^* w_{q+1} = \psi^T f \tag{65}$$

To minimize $r^{(q+1)}$, we could seek ψ so that Δr points in a direction opposite that of $r^{(q)}$. Now, Eq. (55) shows $r^{(q)}$ is in the column space of P^T (row space of P). Moreover, Eqs. (47) and (56) show that

$$P^2 = P \qquad (P^T)^2 = P^T \tag{66}$$

i.e., that P and P^T are idempotent. This means the full n-dimensional space can be written as a direct sum[8] of the column space of P^T and the null space of P^T (these spaces are not orthogonal complements, since P is not symmetric). To pursue this, apply Eq. (22) when Σ is given by Eq. (57) to conclude

$$\Phi^T r^{(q+1)} = 0 \qquad \Psi^T r^{(q+1)} = 0 \tag{67}$$

With Eqs. (67) and (56), it follows that

$$P^T r^{(q+1)} = r^{(q+1)} \tag{68}$$

and hence that $r^{(q+1)}$ is also in the column space of P^T.

The most interesting situation would involve Eq. (64) and a general $f(t)$; one would be guided by information regarding the frequency content of the loading in the selection of ψ. At this time, we will consider a single forcing frequency ω with the load given by Eq. (34). Then, by Eq. (65),

$$w_{q+1} = \frac{\Psi^T f}{(\lambda^* - \lambda)}, \qquad (\lambda = \omega^2) \tag{69}$$

Substituting Eq. (69) into Eq. (64) yields

$$\Delta r = \frac{-(K-\lambda M)\Psi\Psi^T f}{(\lambda^* - \lambda)} \tag{70}$$

Now, recall Ψ will be generated from some Ψ^* by the following steps. Applying Eq. (56) to Eq. (29)

$$\Psi^{**} = P\Psi^*$$

$$\Psi = \Psi^{**}/[\Psi^{**T} M \Psi^{**}]^{1/2} \tag{71}$$

Using Eqs. (62), (70), and (71) produces

$$\Delta r = \frac{-(K-\lambda M)P\Psi^*(P\Psi^*)^T f}{(P\Psi^*)^T (K-\lambda M)(P\Psi^*)} \tag{72}$$

From Eqs. (55), (58), and (72), it follows that

$$r^{(q+1)} = r^{(q)} - \frac{(K-\lambda M)P\Psi^*\Psi^{*T} r^{(q)}}{(P\Psi^*)^T (K-\lambda M)(P\Psi^*)} \tag{73}$$

Observe a choice such that $\psi^{*T} r^{(q)} = 0$ leads to no improvement in the accuracy of analysis. We will thus consider

$$\Psi^{*T} r^{(q)} \neq 0 \tag{74}$$

and seek a condition that will force $r^{(q+1)}$ to vanish. First, note directly from Eq. (56)

$$MP = M - M\Phi\Phi^T M = P^T M = (MP)^T \tag{75}$$

Thus, with Eq. (66)

$$P^T MP = P^T (P^T) M = P^T M = MP \tag{76}$$

Similarly, Eqs. (39), (49), and (56) yield

$$P^T KP = KP = P^T K = (KP)^T \tag{77}$$

Using Eqs. (76) and (77) in Eq. (73) and adopting Eq. (74) allows us to write

$$r^{(q+1)} = r^{(q)} - \frac{(K-\lambda M)P\Psi^*}{\Psi^{*T}(K-\lambda M)P\Psi^*/\Psi^{*T} r^{(q)}} \tag{78}$$

Immediately from Eq. (78), we see that the residual vector can be made to vanish in the $(q+1)$-dimensional approximation (i.e., we can force $r^{(q+1)} = 0$) by requiring

$$(K-\lambda M)P\Psi^* = r^{(q)} = P^T f \tag{79}$$

To explore Eq. (79), note that Eq. (77) yields

$$K^{-1}P^T = PK^{-1} \tag{80}$$

Thus, with Eq. (80), we can write Eq. (79) as

$$(I-\lambda K^{-1}M)P\Psi^* = PK^{-1} f \tag{81}$$

Equation (81) pertains to a single forcing frequency and the general case requires knowledge of the frequency content of the loading. Nevertheless, Eq. (81) is interesting since it provides, at least heuristically, support for use of the Ritz vector approach.[1,2,4] To see this, consider a solution of the form

$$\psi^* = v_0 + \lambda v_1 + \lambda^2 v_2 + \ldots \tag{82}$$

Substituting Eq. (82) into Eq. (81) and equating like powers of λ yields

$$Pv_0 = PK^{-1} f, \quad Pv_1 = K^{-1} MPv_0, \quad Pv_2 = K^{-1} MPv_1 \tag{83}$$

The right-hand sides can be rearranged, noting that Eq. (76) yields

$$K^{-1}P^T MP = K^{-1} MP$$

and thus applying Eq. (80)

$$K^{-1} MP = PK^{-1} MP \tag{84}$$

Using Eq. (84), we see Eq. (83) is satisfied if

$$v_0 = K^{-1} f, \quad v_1 = K^{-1} MPv_0, \quad v_2 = K^{-1} MPv_1 \tag{85}$$

The right-hand sides of Eq. (85) are rather like the vectors generated in the Ritz procedure.[1,2,4] In fact, for $q=0$ (no eigenvectors in the initial approximation), we have $P=I$ by Eq. (56) and then Eq. (85) reduces exactly to vectors generated by Wilson and his colleagues.[1,2,4] More generally, Eq. (85) leads to the same basis when M-orthogonality is applied [e.g., Eqs. (29) and (32)] as do equations of the type of Eq. (31).

If one needs simply the response for a single forcing frequency, this procedure and, indeed, mode superposition are unnecessary. The solution is generated easily from $(K-\lambda M)^{-1}$ and if a power series expression is desired convergence properties are well known.[9,10] However, the simplicity of the result [Eqs. (64) and (65)] for a general forcing function is quite remarkable. Additional numerical experiments are in progress to further test the concept.

Discussion

In this paper, "exact" eigenvectors and eigenvalues mean simply a partial solution to the direct eigenproblem for K and M, obtained by whatever means the user thinks prudent. We recognize errors will occur when solving for the lowest q eigenvalues and eigenvectors for n-dimensional matrices. Our concern is whether, having q such eigenpairs, obtained with whatever precision the user thinks appropriate, there is an easy, efficient way of producing a basis that will be substantially more accurate in handling specific time-dependent design loads than the q-dimensional basis of eigenvectors. It appears that there is such a basis and that it can be formed using Ritz vectors derived from the specific design loads.

In particular, the present work shows that in situations where the q lowest eigenvalues and eigenvectors are available and dynamic analysis "within this frequency range" is required, it would be prudent to augment the q-dimensional basis by two Ritz vectors for each spatial loading case. Observe that the residual vector [Eq. (21)] can be calculated on conclusion of the analysis to see if the desired accuracy has been obtained.

A reviewer with considerable experience in calculations of this type generously made the following helpful observations with which the author concurs, based on expressions for the residual vector presented in this manuscript and the author's rather more limited experience with related numerical experiments:

1) The second term in the expression for the residual [Eqs. (26) and (35)] involves matrix multiplication and thus may be small even if the matrix $[K\Phi^{\#} - M\Phi^{\#}\Lambda^*]$ has components that are large. Thus, the residual vector could be small in magnitude even if the approximation space does not lead to accurate eigensolutions. (Our desire to obtain accurate eigensolutions relates to recovering sufficient conditions for a small residual; they are not necessary.)

2) The computer cost to calculate Ritz vectors is much less than the cost of a corresponding eigensolution based on

subspace iteration or other methods available using MSC/NASTRAN. Thus, a much higher-dimensional approximation space may be generated based on Ritz vectors then would result for the same computer cost with an analysis based on recovering (exact) eigenvectors. Also, the approximate eigenvalues obtained from the residual stiffness [Eq. (6)] are generally more accurate the larger the dimension of the approximation space. Hence, one rather efficient way of checking the accuracy of approximate eigensolutions obtained from Eqs. (6) and (9) would be simply to add additional Ritz vectors to the approximation space Σ to see if previously obtained approximate eigensolutions are again recovered.

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